Necessary Conditions for Solitaire Feasibility

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Abstract.

The classical game of peg solitaire has uncertain origins, but was certainly popular by the time of Louis XIV, and was described by Leibniz in 1710. One of the classical problems concerning peg solitaire is the feasibility issue. An early tool used to show the infeasibility of various peg games is the *rule-of-three* [Suremain de Missery 1841]. In the 1960s the description of the *solitaire cone* [Boardman and Conway] provides necessary conditions: valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. In this paper, we recall these necessary conditions and present new developments: the *lattice criterion*, which generalizes the rule-of-three; and results on the strongest pagoda functions, the facets of the solitaire cone.

1 Introduction and Basic Definitions

1.1 Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes - see Fig. 1 - although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column respectively) two consecutive pegs are adjacent to an empty hole in the same row (column respectively), we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of Beasley [5]. The book mentions an engraving of Berey, dated 1697, of a lady with a solitaire board. The book also contains a quotation of Leibniz [9] which was written for the Berlin Academy in 1710. Apparently the first theoretical study of the game that was published was done in 1841 by Suremain de Missery, and was reported in a paper by Vallot [13]. The modern mathematical study of the game dates to the 1960s at

This paper in an extended version of [4]

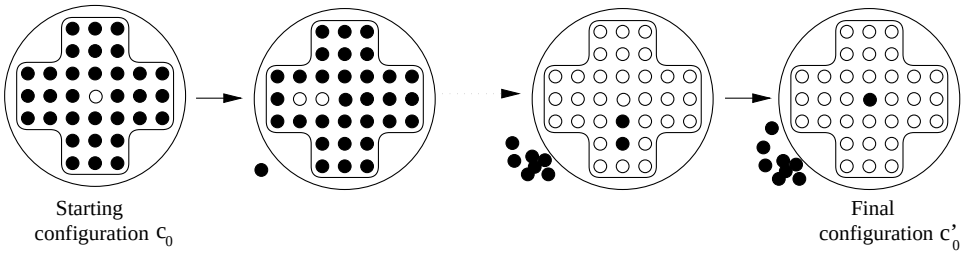

Figure 1: A feasible English solitaire peg game with possible first and last moves

Cambridge University. The group was led by Conway who has written a chapter in [6] on various mathematical aspects of the subject. One of the problems studied by the Cambridge group is the following basic *feasibility* problem (see Definition 1.1 in the sequel for a formal definition):

Peg solitaire feasibility problem. Given a board B and a pair of configurations (c, c') on B, determine if the pair (c, c') is *feasible*, that is, if there is a legal sequence of moves transforming c into c'.

The complexity of the feasibility problem for the game played on a n by n board was shown by Uehara and Iwata [12] to be NP-complete, so easily checked necessary and sufficient conditions for feasibility are unlikely to exist. In this paper, we recall constructions used to prove the infeasibility of some pair (c, c'): the *rule-of-three* in Section 2, the *solitaire cone* in Section 3.1; and present new developments in Section 3.2 and Section 4.

1.2 Basic definitions

In this section we introduce some terminology used throughout this paper.

The *board* of a peg solitaire game is a *finite* subset $B \subset \mathbb{Z}^2$. of the lattice of integer points in the plane. Thus, B stands for the set of locations (i, j) of holes of the board on which the game is played. For example, the classical 33-board is:

$$B = \{(i,j) : \ -1 \le i \le 1, \ -3 \le j \le 3\} \cup \{(i,j) : \ -3 \le i \le 3, \ -1 \le j \le 1\}.$$

A *configuration* c on the board is an integer vector $c \in \mathbb{Z}^B \subset \mathbb{R}^B$. It can be interpreted as a configuration of pegs on the board: in the usual game, all configurations c lie in $\{0,1\}^B$, with the interpretation that hole $(i, j) \in B$ contains a peg if $c_{i,j} = 1$ and is empty if $c_{i,j} = 0$; extending this, we allow any integer (possibly negative) number $c_{i,j}$ of pegs to occupy any hole $(i, j) \in B$.

The *complement* of a $\{0,1\}$-configuration $c \in \{0,1\}^B$ is defined to be the configuration $\bar{c} := \mathbb{1} - c$ where $\mathbb{1} = (1, 1, \ldots, 1) \in \mathbb{R}^B$ is the all-ones configuration.

For each $(i, j) \in B$ let $e_{i,j}$ be the (i, j)th unit vector in $\mathbb{R}^B$, which is in particular the configuration with a unique peg in the (i, j)th position. A *move* or a *jump* μ is a vector in $\mathbb{R}^B$ which has 3 non-zero entries: two entries of -1 in the positions from which pegs are

removed and one entry of 1 for the hole receiving the new peg. We can now make the peg solitaire feasibility problem precise.

Definition 1.1 Given a board B and an associated set of moves $\mathcal{M}$, a pair (c, c') of configurations is *feasible* if there is a sequence $\mu^1, \ldots, \mu^k \in \mathcal{M}$ of moves on B such that

$$c' - c = \sum_{i=1}^{k} \mu^i \text{ and } c + \sum_{j=1}^{i} \mu^j \in \{0,1\}^B \text{ for } i = 1, \ldots, k$$

For instance, the English 33-board admits 76 moves; see Fig. 1 for possible first and last moves in some sequence of moves transforming the initial configuration c_0 to its complementary c_0'.

2 The Rule-of-Three

In this section we recall the so-called rule-of-three (cf. [5, 6]), a classical construction used to test solitaire game feasibility. The rule-of-three can be used, for example, to show that on the cross shaped English 33-board, starting with the initial configuration c_0 of Fig. 1, the only reachable final configurations with *exactly one* peg are c_0' (see Fig. 1), c_1', c_2', c_3' and c_4' (see Fig. 2) with, respectively, a final peg in position $(0,0), (-3,0), (0,3), (3,0)$ and $(0,-3)$.

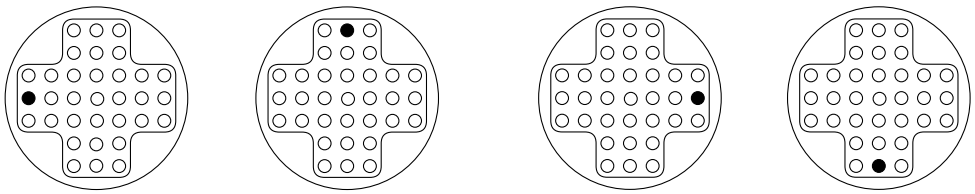

Figure 2: Four other feasible final configurations

Let $\mathbb{Z}_2 := \{a, b, c, e\}$ be the Abelian group with identity e and addition table

$$a + a = b + b = c + c = e, \quad a + b = c, \ a + c = b, \ b + c = a.$$

Define the following two maps $g_1, g_2 : \mathbb{Z}^2 \longrightarrow \mathbb{Z}_2$, which simply color the integer lattice $\mathbb{Z}^2$ by diagonals of a, b and c in either direction; see Fig. 3:

$$g_1(i,j) := \begin{cases} a & \text{if } (i+j) \equiv 0 \pmod 3 \\ b & \text{if } (i+j) \equiv 1 \pmod 3 \\ c & \text{if } (i+j) \equiv 2 \pmod 3 \end{cases}, \qquad g_2(i,j) := \begin{cases} a & \text{if } (i-j) \equiv 0 \pmod 3 \\ b & \text{if } (i-j) \equiv 1 \pmod 3 \\ c & \text{if } (i-j) \equiv 2 \pmod 3 \end{cases}.$$

For any board $B \subset \mathbb{Z}^2$, define the *score map* to be $\mathbb{Z}$-module homomorphism

$$\begin{array}{ccccc} \phi & : & \mathbb{Z}^B & \longrightarrow & \mathbb{Z}_2^2 \\ & & e_{i,j} & \mapsto & (g_1(i,j) \ , \ g_2(i,j)) \end{array} .$$

Thus, the score of a configuration $c \in \mathbb{Z}^B$ is given by

$$\phi(c) \quad = \quad \sum_{(i,j)\in B} c_{i,j} \cdot (g_1(i,j)\ ,\ g_2(i,j)) \quad .$$

Since the board B under discussion will always be clear from the context, we use the notation ϕ for any board. For instance, the score of the configuration $c'_0 = e_{0,0}$ of one peg in the center of the English 33-board is $\phi(c'_0) = (a, a)$, as is also the score of its complement $c_0 = \mathbb{1} - e_{0,0}$; see Fig. 3. The score of the board B (all holes filled) is defined to be

$$\phi(B) \quad := \quad \phi(\mathbb{1}) \quad = \quad \sum_{(i,j)\in B} \phi(e_{i,j}) \quad = \quad \sum_{(i,j)\in B} (g_1(i,j)\ ,\ g_2(i,j)) \quad .$$

Note that $\phi(B) = \phi(c) + \phi(\bar{c})$ for any configuration c and its complement $\bar{c}$. Thus, the score of the English 33-board is $\phi(B) = \phi(e_{0,0}) + \phi(\mathbb{1} - e_{0,0}) = (a, a) + (a, a) = (e, e)$.

It is easy to verify that any feasible move μ on any board B has the identity score $\phi(\mu) = (e, e)$. This gives the following proposition.

Proposition 2.1 [The rule-of-three]
A necessary condition for a pair of configuration (c, c') to be feasible is that $\phi(c' - c) = (e, e)$, namely, $c' - c \in Ker(\phi)$.

Using Proposition 2.1, we can show that, besides the configuration c'_0 given in Fig. 1, the only final configuration c' with exactly one non-zero entry $c'_{i,j} = 1$ forming a feasible pair (c_0, c') are the 4 configurations c'_1, c'_2, c'_3 and c'_4. Fig. 3 shows that $\phi(c') = (a, a) = \phi(c_0)$ if c' is one of $c'_0, c'_1 \ldots c'_4$, whereas $\phi(c') \neq (a, a)$ otherwise.

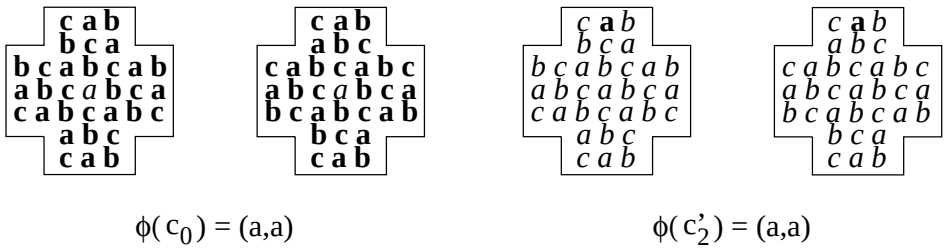

Figure 3: The score of a final configuration with only one peg remaining

With $\bar{c} = \mathbb{1} - c$ the complement of c, the rule-of-three implies that $(c, \bar{c})$ is feasible only if $\phi(\bar{c}) = \phi(c)$, which is equivalent to $\phi(B) = \phi(c) + \phi(\bar{c}) = \phi(c) + \phi(c) = (e, e)$. In other words:

Proposition 2.2 *Let B be any board. A necessary condition for the configurations pair $(c, \bar{c})$ to be feasible, with $\bar{c} = \mathbb{1} - c$ the complement of c, is that the board score is $\phi(B) = (e, e)$.*

An application of Proposition 2.2 is that complementary games are not feasible on the following *French* and *Thai* boards given in Fig. 4 since in both cases $\phi(B) = (a, a)$. A Board B satisfying $\phi(B) = (e, e)$, such as the English 33-board, is called a *null-class* board in [5].

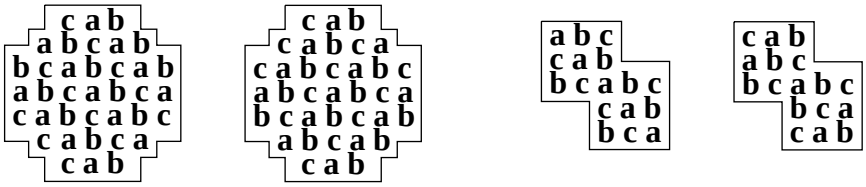

Figure 4: Two non null-class boards

3 Solitaire Cone and Pagoda Functions

3.1 Solitaire cone

A first relaxation of the feasibility problem is to allow any integer (positive or negative) number of pegs to occupy any hole for any *intermediate* configurations. We call this game the *integer game*, and call the original game the *0-1 game*. Note that in a 0-1 game we require that for each intermediate configuration of the game a hole is either empty or contains a single peg. Clearly:

$$\text{c',c is } \textit{integer} \text{ feasible} \iff c' - c \in IC_B = \Big\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu \quad : \lambda_\mu \in \mathbb{N} \Big\}$$

where the *integer solitaire cone* IC_B is the set of all *non-negative integer* linear combinations of moves. Unfortunately deciding if $c' - c$ can be expressed as the sum of move seems to be a hard computational problem. We get the following necessary criterion:

Proposition 3.1 [The integer cone criterion]
A necessary condition for a pair of configurations (c, c') *to be feasible is that* $c' - c \in IC_B$.

A further relaxation of the game leads to a more tractable condition. In the *fractional game* we allow any fractional (positive or negative) number of pegs to occupy any hole for any *intermediate* configurations. A *fractional move* is obtained by multiplying a move by any positive scalar and is defined to correspond to the process of adding a move to a given configuration. For example, let c =[1 1 1], c' =[1 0 1]. Then $c' - c$ =[0 -1 0]=$\frac{1}{2}$[-1 -1 1]+$\frac{1}{2}$[1 -1 -1] is a feasible fractional game and can be expressed as the sum of two fractional moves, but is not feasible as a 0-1 or integer game. Clearly:

$$\text{c',c is } \textit{fractional} \text{ feasible} \iff c' - c \in C_B = \Big\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu \quad : \lambda_\mu \in \mathbb{R}^+ \Big\}$$

where the *solitaire cone* C_B is the set of all *non-negative* linear combinations of moves. We get the weaker, but useful, following necessary criterion:

Proposition 3.2 [The cone criterion]
A necessary condition for a pair of configurations (c, c') *to be feasible is that* $c' - c \in C_B$.

The condition $c' - c \in C_B$ provides a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for C_B which is violated by $c' - c$. According to [5], page 71, these inequalities "were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway...". They are also known as *resource counts*, and are discussed in some detail in Conway [6]. The strongest such inequalities are induced by the facets of C_B. For example, the facet (iii) (given by Beasley) of Fig. 6 induces an inequality $a \cdot x \leq 0$ that is violated by $c' - c$ with (c, c') given in Fig. 5: $(c' - c) \cdot a = 2 > 0$. This implies that this game is not feasible even as a fractional game and, therefore, not feasible as an integer game or classical 0-1 game either.

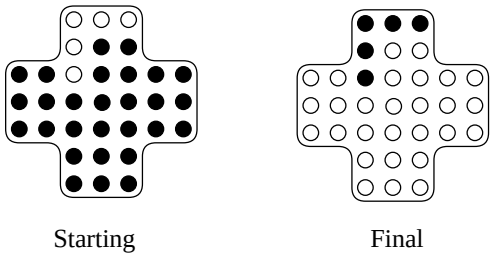

Figure 5: An infeasible classical solitaire peg game

3.2 Facets of the solitaire cones

Most of the results can be applied to boards which are subsets of the square lattice in the plane, such as the original peg solitaire board. For simplicity let us consider rectangular boards. For $n \geq 4$ or $m \geq 4$, the solitaire cone $C_{B_{m,n}}$ associated to the m by n board $B_{m,n}$ is a pointed full-dimensional cone and the moves of the solitaire cone are extreme rays; see [2] for a detailed study of $C_{B_{m,n}}$. Let us assume that $m \geq 4$ or $n \geq 4$ and that the holes of $B_{m,n}$ are ordered in some way. It is convenient to display $c' - c$ and μ as m by n matrices, although of course all products should be interpreted as dot products of the corresponding mn-vectors. We represent the coefficients of the facet inducing inequality $az \leq 0$ by the m by n array $a = [a_{i,j}]$. It is a convenient abuse of terminology to refer to a as a *facet* of C_B. A *corner* of a is a coefficient $a_{i,j}$ with $i \in \{1, m\}$ and $j \in \{1, n\}$. The notation $\mathcal{T} = (t_1, t_2, t_3)$ refer to a *consecutive triple* of row or column indices. Each consecutive triple defines a *triangle inequality*: $a_{t_1} \leq a_{t_2} + a_{t_3}$ and a triangle inequality is *tight* if equality holds. The following theorem summarizes known results on properties of valid inequalities (pagoda functions) for C_B.

Theorem 3.3 [5] *For each valid inequality $a = [a_{i,j}]$ for C_B*

1. *For every consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$, the associated triangle inequality must hold.*
2. *Negative coefficients of a can only occur in corners.*
3. *If $\mathcal{T} = (t_1, t_2, t_3)$ is a consecutive triple with $a_{t_2} = 0$ then $a_{t_1} = a_{t_3}$.*

4. *If two consecutive row (respectively, column) entries of a are zero the entire row (respectively, column) is zero.*

It is not feasible to generate all facets for reasonably sized boards, and in general no characterization of facets is known. A large class of facets can, however, be generated by the following procedure.

Genfacet /*procedure to generate a facet matrix a of C_B */

1. Choose a proper subset of coefficients of a satisfying: (a) If a corner is chosen, all coefficients in the row and/or column of at length at least 4 containing the corner must also be chosen; and (b) If two consecutive coefficients are chosen, their entire row and column must also be chosen. Set these chosen coefficients to zero.

2. Choose any undefined coefficient that is not a corner and set it to one.

3. Choose a consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ for which precisely two of the corresponding coefficients of a are defined. Define the remaining coefficient by the associated tight triangle inequality providing this does not violate any other triangle inequality for a.

4. Repeat step 3 until no further coefficient of a can be defined.

Theorem 3.4 [2] *Given an m by n board B, with $m \geq 4$ or $n \geq 4$, if Genfacet terminates with all elements of a defined, then a is a facet of C_B.*

Genfacet can easily be adapted to non-rectangular boards that are connected subsets of the square grid, such as the original peg solitaire game. The notion of corner generalizes in the obvious way to all holes that have exactly one horizontal and vertical neighbour. For example, the original English game has 8 corners and by Genfacet we can generate the facets (i) and (ii), but not facet (iii), given in Fig. 6.

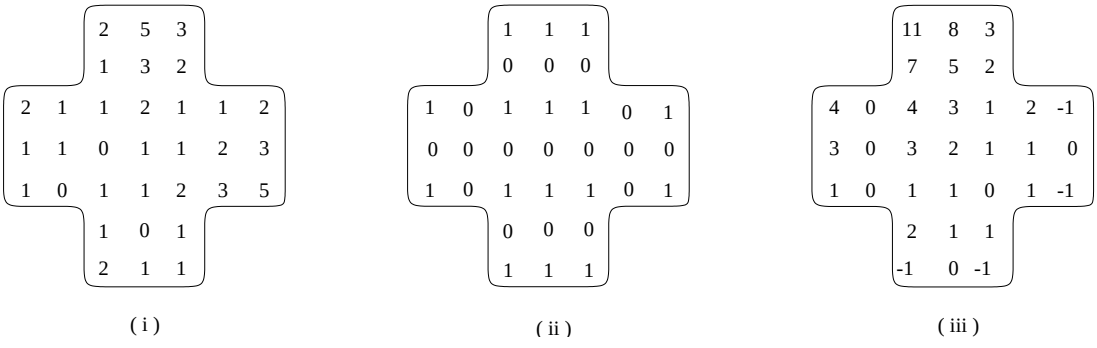

Figure 6: Three facets of the English solitaire cone

Remark 3.5

1. *If condition (a) in step 1 is dropped, Genfacet could generate matrices that are not facets. For example, the matrices*

$$a = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 1 \end{pmatrix}$$

   have the same tight triangle inequalities. Since one is not a scalar multiple of the other, they are not facets. However a would be generated by Genfacet if condition (a) in Step 1 is dropped.

2. *Not all facets of rectangular boards are generated by Genfacet. For example the following facet (found by computer) of the 3 by 4 rectangular board cannot be generated by Genfacet.*

$$\begin{pmatrix} -1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

3. Conway *[6] gives 14 valid inequalities (pagoda functions) for the English solitaire game. Of these, 11 described in his Figures 21(d), 22(a)-(d) are facets of C_B and can be generated by Genfacet. The two pagoda functions in Figures 22(h) and 22(v) - see [6] - are not facets since (h) is one half the sum of the inequality from (a) (interchanging the 0 and 1 in (a) as shown) and the same inequality with -1 replaced by +1, similarly, (v) is half the sum of 2 inequalities derived from (b) again with the indicated interchanges. The pagoda function in Figure 21(c) - see Fig. 7 - is a facet of C_B, and is not generated by Genfacet.*

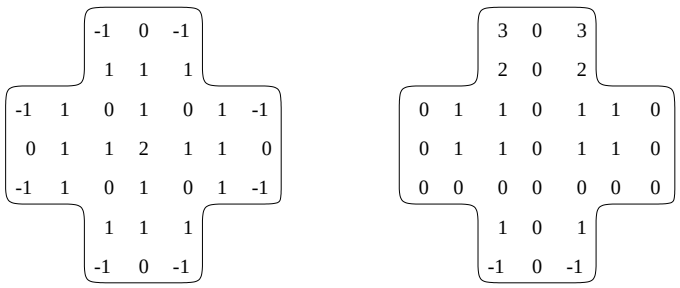

Fig. 21 (c) Fig. 21 (d)

Figure 7: Two pagoda function from Conway's book [6]

We recall some interesting applications of Genfacet including a characterization of 0-1 facets, see Theorem 3.6, and exponential upper and lower bounds on the number of facets,

see Lemma 3.8 and Theorem 3.9, (in the dimension $|B|$ of C_B), see [2] for details and other geometric and combinatorial properties of C_B.

Using Theorems 3.3 and 3.4, we obtain a complete characterization of 0-1 facets of C_B when B is a toric board. Let α be an m by n 0-1 matrix. We define the a graph G_α on α as follows: vertices of G_α correspond to non-zero coefficients, and two vertices are adjacent if the corresponding coefficients are in some consecutive triple where the remaining coefficient is zero. Note that in fact there must be at least two such triples since if (t_1, t_2, t_3) is such a triple then so is (t_3, t_2, t_1).

Theorem 3.6 [characterization of 0-1 valued facets]
Let B be the m by n toric board. A m by n 0-1 matrix α is a facet of C_B if and only if

1. *No non-zero row or column contains two consecutive zeroes, and*
2. *G_α is connected.*

1	0	1	1
0	0	0	0
1	0	1	1
0	0	0	0

1	0	1	0
0	1	0	1
1	0	1	0
0	1	0	1

Figure 8: Two pagoda functions of $C_{4\times 4}$, only the first one being a facet

Theorem 3.6 is useful for proving large classes of 0-1 matrices are facets. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be two vectors. We say the m by n matrix a is the *product* of x and y if for all $1 \leq i \leq m$ and $1 \leq j \leq n$ $a_{i,j} = x_i y_j$. A simple application of Theorem 3.6 gives the following:

Corollary 3.7

1. *A 0-1 n-vector is a facet of the 1 by n toric board if and only if it has no pair of consecutive zeroes, no string of five or more ones, and at most one string of four ones.*
2. *The product of two 0-1 facets of the 1 by m and 1 by n toric boards gives a 0-1 facet of the m by n toric board.*

As mentioned earlier, experimental evidence and the fact that the n by n game is NP-complete indicates that solitaire cones are likely to have a very large number of facets. In the following, we give some additional theoretical evidence for this observation.

Lemma 3.8 [upper bound]
Let $|\mathcal{M}|$ be the number of moves on a board B with $|B|$ holes. C_B has at most $\binom{|\mathcal{M}|}{|B|-1} < 2^{|\mathcal{M}|}$ facets. In particular the m by n toric board generates at most $\binom{4mn}{mn-1} < (\frac{4^4}{3^3})^{mn}$ facets.

Theorem 3.9 [lower bound]

1. *There are at least* $6^{\frac{m+n}{9}}$ *0-1 facets of the* m *by* n *toric board generated by products of facets of the 1 by* m *and 1 by* n *toric boards, for* m *and* n *both divisible by 9.*

2. *There are at least* $2^{\frac{mn}{4}}$ *0-1 facets generated by the* m *by* n *toric board, for even values of* $m \geq 4$*, and* n *divisible by 4.*

Finding all facets of the cone C_B is an example of a *convex hull* or *vertex enumeration* problem, for which various computer programs are available. The computational results in [2] were obtained using the double description method *cdd* implemented by FUKUDA [8], and the reverse search method *lrs* implemented by AVIS [1]. We made use of these codes to completely generate all facets for some small boards (such as the 95 444 facets for the toric closure of the 4 by 4 board, see [2]). For realistically sized boards the corresponding solitaire cones are too large for these programs. For example the original peg solitaire game gives rise to a cone with 76 extreme rays in 33 dimensions. The reverse search vertex enumeration code *lrs* applied to this cone generated over 300,000 facets from about 6 million bases of the cone that were generated before the program was terminated. An unbiased estimate obtained from *lrs* suggests that the solitaire cone has about 9.2 million facets and 12 billion bases. If correct, this estimate puts the size of the cone just beyond problems currently solvable by a parallel version of *lrs*.

We end this section by remarking that a 0-1 vector a is valid over C_B if and only if the position $c' = a$ is not reachable from any other position in the 0-1 peg game. To see this note that if $c' = a$ is reachable, then the last jump must result in the configuration 0 0 1 in some row or column, violating the triangle inequality, so a is not valid. Conversely if a is not valid, it must contain the string 0 0 1 in some row or column. Replacing this by the string 1 1 0 gives a position leading to $c' = a$. A similar statement is not true for the integer game: for example the valid $a = 0\ 1\ 1$ for the 1 by 3 game can be reached from 1 2 0. It would be interesting to see if the 0-1 facets of the 0-1 peg game have some natural interpretation in terms of the game itself.

4 The Lattice Criterion

4.1 The solitaire lattice

For the fractional game, we relaxed the integrality while keeping the non-negativity condition. Another relaxation of the integer game is to drop the non-negativity while keeping the integrality. It amounts, besides allowing any integer (positive or negative) number of pegs for any intermediate configurations, to allow *additive moves*. The configuration of an additive move μ^+ (jumping over an empty hole and putting a peg in) is $c_\mu^+ = -c_\mu$ where c_μ is the configuration of an ordinary (subtractive) move μ. We call this game the *lattice game*. Clearly:

$$\text{c',c is } \textit{lattice} \text{ feasible} \iff c' - c \in L_B = \Big\{ \sum_{\mu \in \mathcal{M}} \lambda_\mu \mu \quad : \lambda_\mu \in \mathbb{Z} \Big\}$$

where the *solitaire lattice* L_B is the set of all *integer* linear combinations of moves. It gives the following criterion (weaker than Proposition 3.1):

Proposition 4.1 [The lattice criterion]
A necessary condition for a pair of configurations (c, c') *to be feasible is that* $c' - c \in L_B$.

Since the score ϕ is a homomorphism of $\mathbb{Z}$-modules which maps each lattice generator $\mu \in \mathcal{M}$ to (e, e), it follows that $\phi(v) = (e, e)$ for any $v \in L_B$; i.e., for any board B and any pair (c, c') on B if $c' - c \in L_B$ then $c' - c \in \mathrm{Ker}(\phi)$. In other words, as stated in the following proposition, the lattice criterion is generally stronger than the rule-of-three.

Proposition 4.2 *For any board* B, *we have* $L_B \subseteq Ker(\phi)$.

Fig. 9 provides an example of a null-class board and a game on it whose associated pair $(c, \bar{c})$ satisfies $\bar{c} - c \in \mathrm{Ker}(\phi)$ but $\bar{c} - c \notin L_B$ (see Corollary 4.6). This shows that the lattice criterion may be strictly stronger than the rule-of-three and therefore could be more useful in proving infeasibility.

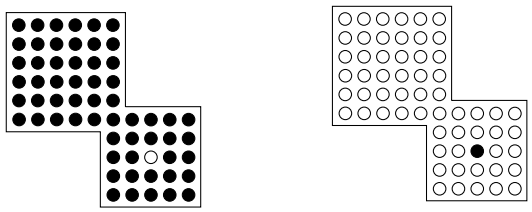

Figure 9: An infeasible game satisfying the rule-of-three but not the solitaire lattice criterion

4.2 The lattice criterion versus the rule-of-three

We proceed to discuss the relations between the lattice criterion and the rule-of-three, namely, between L_B and $\mathrm{Ker}(\phi)$. Any lattice $L \subseteq Z^d \subset \mathbb{R}^d$ has a *basis*, that is, a free generating set. Any basis is also $\mathbb{R}$-linearly independent. The *rank* of the lattice is the cardinality of any basis, and the lattice is *full rank* if its rank is d, namely, the $\mathbb{R}$-span of the lattice is the entire space $\mathbb{R}^d$. The solitaire lattice of a board B is typically of full rank $|B|$. The *determinant* of a full rank lattice $L \subseteq \mathbb{R}^d$ is defined to be the absolute value $\det(L) := |det(V)|$ of the determinant of any (ordered) basis $V = [v_1, \ldots, v_d]$ of L. Since $L_B \subseteq \mathrm{Ker}(\phi)$ for any board B, we always have $\mathbb{Z}^B/\mathrm{Ker}(\phi) \subseteq \mathbb{Z}^B/L_B$ and, therefore, $|\mathbb{Z}^B/\mathrm{Ker}(\phi)| \leq |\mathbb{Z}^B/L_B|$. For a typical board B, the map ϕ is onto and the lattice L_B is full rank, giving

$$|\mathbb{Z}^B/\mathrm{Ker}(\phi) = |\mathrm{Im}(\phi)| = |\mathbb{Z}_2^2| = 16$$

and

$$|\mathbb{Z}^B/L_B| = \det(L_B).$$

This gives the following useful lemma.

Lemma 4.3 *For a board B such that ϕ is an onto map and L_B is full rank, $L_B = Ker(\phi)$ if and only if* $\det(L_B) = 16$.

For example, the board given in Fig. 9, produces a full rank lattice L_B and an onto map ϕ but $\det(L_B) = 32$ - see Corollary 4.6 - and therefore L_B is strictly contained in $\mathrm{Ker}(\phi)$. The (complementary) game of Fig. 9 is infeasible while satisfying the rule-of-three.

The computation of $|\mathbb{Z}^B/\mathrm{Ker}(\phi)|$ is generally easy: One can easily check that the 16 possible $\{0,1\}$-configurations on a 2×2 board are mapped by ϕ precisely onto the 16 elements of $\mathbb{Z}_2^2$. In other words, if a board B contains a 2×2 sub-board then $|\mathbb{Z}^B/\mathrm{Ker}(\phi)| = |\mathrm{Im}(\phi)| = 16$.

Proposition 4.4 *If a board B contains a 2×2 sub-board then $|\mathbb{Z}^B/Ker(\phi)| = |Im(\phi)| = 16$.*

4.3 The lattice criterion beats the rule-of-three

Let B be a board and let $V = \{v_1, \ldots, v_d\}$ be a basis of the solitaire lattice L_B. Then for any two given configurations c, c' on B, the necessary condition $c' - c \in L_B$ for feasibility provided by Proposition 4.1 holds if and only if the linear system $\sum_{i=1}^{d} \lambda_i v_i = c' - c$ is solvable (over $\mathbb{R}$) and its unique solution is an integer vector $\lambda \in \mathbb{Z}^d$. This can be efficiently tested, say by Gaussian elimination. Thus, in order to efficiently facilitate the lattice criterion for game feasibility, we need to determine a basis of L_B. One way to find a basis is by applying the *Hermite normal form algorithm* (cf. Schrijver [11]) to any generating set of the lattice. However, it is more efficient and illuminating to *characterize* a canonical basis for the lattice whenever possible. This approach is taken, for instance, in Lovasz' outstanding work [10] on the *matching lattice*. A close study of $L_{B_{m,n}}$ provides a canonical basis for this lattice. In fact, this basis is precisely the *Hermite basis* of the lattice which would have been produced by the Hermite normal form algorithm. However, this characterization allows to write it down at once, without applying the time consuming Hermite normal form algorithm; see [7] for details. We have:

Theorem 4.5 *Let $B_{m,n}$ be any $m \times n$ board with $n \geq 4$ or $m \geq 4$. Then the following hold for the solitaire lattice $L_{B_{m,n}}$:*

1. *The solitaire lattice $L_{B_{m,n}}$ is characterized by:*

$$c \in L_{B_{m,n}} \iff \begin{cases} \sum\{c_{i,0} \;:\; 0 \leq i \leq n-1,\ i \not\equiv 0 \ (mod\ 3)\ \} & \equiv 0 \ (mod\ 2) \\ \sum\{c_{i,0} \;:\; 0 \leq i \leq n-1,\ i \not\equiv 1 \ (mod\ 3)\ \} & \equiv 0 \ (mod\ 2) \\ \sum\{c_{i,1} \;:\; 0 \leq i \leq n-1,\ i \not\equiv 0 \ (mod\ 3)\ \} & \equiv 0 \ (mod\ 2) \\ \sum\{c_{i,1} \;:\; 0 \leq i \leq n-1,\ i \not\equiv 1 \ (mod\ 3)\ \} & \equiv 0 \ (mod\ 2) \end{cases}$$

2. *$L_{B_{m,n}}$ has full rank with determinant $det(L_{B_{m,n}}) = 16$.*

3. *$|\mathbb{Z}^{B_{m,n}}/Ker(\phi)| = 16 = |\mathbb{Z}^{B_{m,n}}/L_{B_{m,n}}|$ hence $L_{B_{m,n}} = Ker(\phi)$.*

Using the basis of $L_{B_{m,n}}$, one can efficiently compute the (Hermite normal) basis of more complex boards. The algorithm consists in covering a board B by overlapping rectangular sub-boards B_i, $i = 1, \ldots, I$. Then we append each matrix $A^i_{m_i,n_i}$ $i = 1, \ldots, I$ consisting of

the (row and column) moves within each sub-board B_i and add the *cross moves*, that is, the moves from one sub-board to another. Since the number of cross moves is, in general, quite small, the computation of the resulting (Hermite normal) basis of the initial board B is, in general, a bit tedious but easy. This algorithm also provides a rough upper bound for the determinant, that is, $\det(L_B) \leq \prod_{i=1}^{I} det(L_{B_i}) \leq 2^{4I}$. For example, the board B_{60} given in Fig. 9 is made of two overlapping square boards $B_{6,6}$ and $B_{5,5}$ with 4 cross-moves centered on the common hole. It directly gives the following proposition which, in particular, excludes any complementary game $(c, \bar{c})$ while, B_{60} being a null-class board, complementary games satisfy the rule-of-three.

Corollary 4.6

1. *$L_{B_{60}}$ has full rank with determinant $det(L_{B_{60}}) = 32$.*
2. *$|\mathbb{Z}^{B_{60}}/Ker(\phi)| = 16 \neq 32 = |\mathbb{Z}^{B_{60}}/L_{B_{60}}|$ hence $L_{B_{60}} \neq Ker(\phi)$.*
3. *$\mathbb{1} \notin L_{B_{60}}$, that is, any complementary game $\{c, \bar{c}\}$ is infeasible on B_{60}.*

The computation the basis of $L_{B_{60}}$ can be easily extended to any set of k rectangular boards pairwise overlapping on a common corner. The resulting k-hook board - see for example Fig. 10 - will give a lattice with determinant 2^{k+3}. Hook boards demonstrate that the solitaire lattice criterion can be *exponentially finer* than the rule-of-three in the sense of the following theorem.

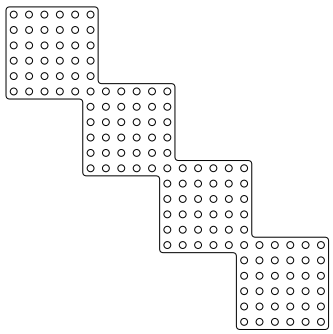

Figure 10: A hook board B with associated lattice satisfying $\det(L_B) = 128$

Theorem 4.7 *For hook boards, the solitaire lattice condition exponentially outperforms the rule-of-three, that is, for every k and every k-hook board B, the ratio of the number of congruence classes of $\mathbb{Z}^B$ modulo L_B to the number of congruence classes of $\mathbb{Z}^B$ modulo $Ker(\phi)$ satisfies*

$$\frac{|\mathbb{Z}^B/L_B|}{|\mathbb{Z}^B/Ker(\phi)|} \quad = \quad 2^{k-1} \quad .$$

Similar computations give that both the classical English 33-board B_{33} and the French 37-board B_{37} satisfy:

Corollary 4.8

1. *$det(L_{B_{33}}) = 16$, hence $L_{B_{33}} = Ker(\phi)$,*
2. *$det(L_{B_{37}}) = 16$, hence $L_{B_{37}} = Ker(\phi)$.*

On the other hand, the following null-class 165-board B_{165} yields to a solitaire lattice satisfying $\det(L_{B_{165}}) = 256$.

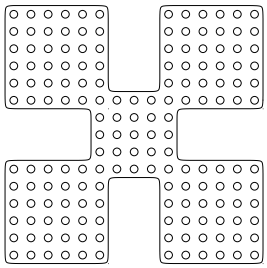

Figure 11: A null-class board B satisfying $\det(L_B) = 256$

5 Conclusion

The solitaire game provides an nice application for classical combinatorial tools such as cone, lattice and integer cone. Checking membership in the lattice L_B is usually easy (once we have a basis) and checking membership in the cone C_B amounts to solve a linear program in polynomial time. Combining these two criteria, that is, checking membership in $C_B \cap L_B$, is usually efficient in proving infeasibility. For example, while the game in Fig. 5 satisfies $c' - c \in L_B$ but $c' - c \notin C_B$, the central game (see Fig. 1) played on a French board (see Fig. 4) satisfies $c' - c \in C_B$ but $c' - c \notin L_B$. Note that for both the French and English boards, we have $L_B = \ker(\phi)$ (see Corollary 4.8); therefore checking the membership in L_B can be easily done using the rule-of-three. The membership in C_B of the central game played on a French board can be shown by moving all pegs on the boundary to the inner part of the board, then moving one peg in the center and finally remove the other pegs using the fractional move given after Proposition 3.1. Clearly, we have $C_B \cap L_B \subset IC_B$ but this inclusion is strict as illustrated by Fig. 12. A further step could be to find a relatively small generating set (Hilbert basis) for the integer cone IC_B of some interesting classes of boards B.

Figure 12: A lattice and fractional feasible but integer infeasible game

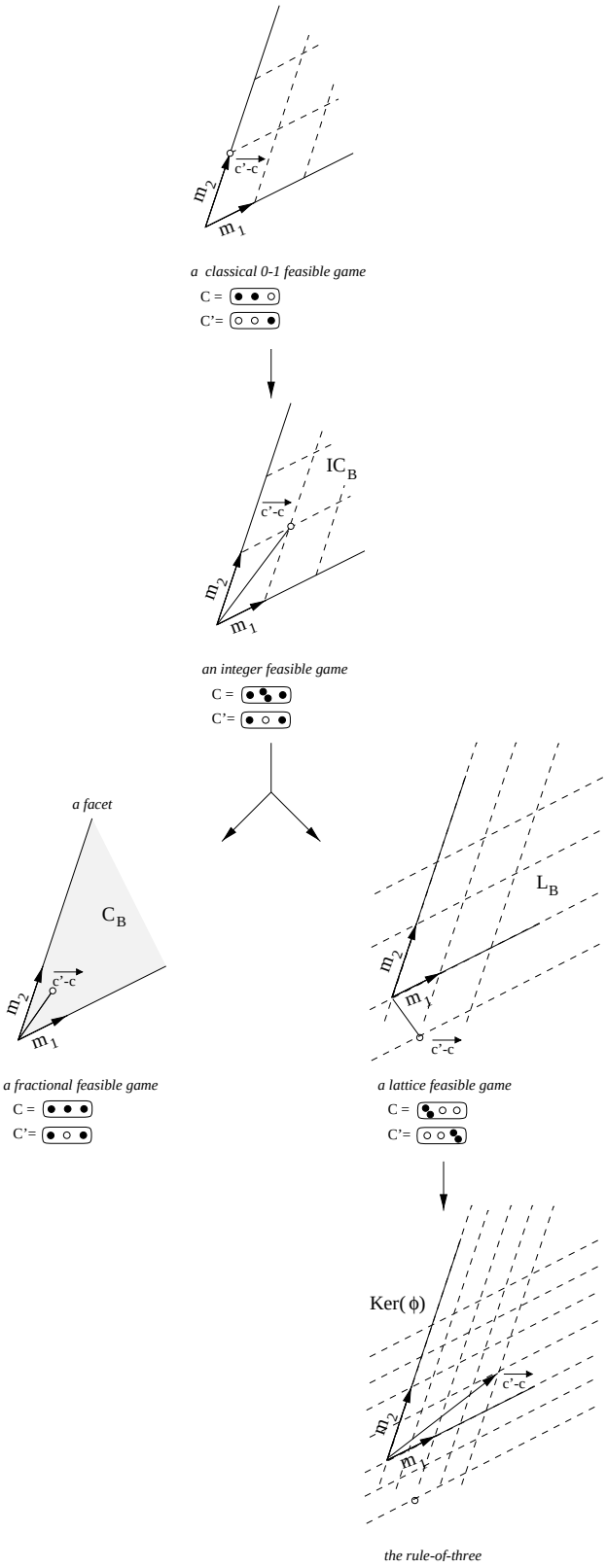

Figure 13: The rule-of-three and the integer, fractional and lattice relaxations

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